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ON THE NATURE OF THE CONTACT STRESS SINGULARITIES UNDER AN ANNULAR STAMP

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The nature of the normal stress singularities under an annular stamp as one approaches the outer and inner contours is clarified.

An approach permitting to obtain an asymptotic expansion for the contact stress which consists of one term (an asymptotic representation in the Erdelyi terminology), is developed. The method proposed permits the investigation of a number of contact problems associated with an annular stamp. However, only an axisymmetric contact problem is considered in this paper. A survey of the research devoted to the problem of impressing an annular stamp into an elastic half-space is presented in [1, 2].

The problem of the behavior of solutions of elasticity theory boundary value problems in the neighborhood of points and lines of separation of boundary conditions was examined in [3 — 8], etc.

1. We use a r, φ, z cylindrical coordinate system, whose z-axis is perpendicular to the half-space boundary. Let an annular stamp with a flat base be located at the boundary of an elastic half-space. A vertical force P directed along the axis of symmetry acts on the stamp. It is assumed that the surface of the half-space is stress-free outside the stamp, and there are no friction forces between the stamp and the elastic half-space.

This problem has been reduced to three integral equations in [9], which can be written briefly as

$$S_{-1/2,1}\,\varphi(r) = f(r), \qquad S_{0,0}\varphi_*(r) = g(r) \tag{1.1}$$

by using the Hankel operator [10]

$$S_{\eta,\alpha} f(x) = 2^{\alpha} x^{-\alpha} \int_0^{\infty} t^{1-\alpha} J_{2\eta+\alpha}(xt) f(t)\, dt$$

For the function $f(r)$ we have

$$f(r) = \begin{cases} f_1(r), & 0 < r < a \\ f_2(r), & a < r < b \\ f_3(r), & b < r < \infty \end{cases}$$

An analogous formula can be written for $g(r)$ as well. In this case we have (the functions f_1, f_3 and g_2 are unknown)

$$g_1(r) = 0 \qquad g_3(r) = 0, \qquad f_2(r) = -2\mu\delta / [(1-\nu)\, r] \tag{1.2}$$

Here μ is the shear modulus, ν is the Poisson's ratio for the material of the half-space, δ is the depth of impression of the stamp, and a and b are the inner and outer radii of the annular stamp.

The normal stress on the contact area $\sigma_z(r, 0)$ is related to the function $\varphi(\xi)$ by the relationship

$$\sigma_z(r, 0) = \int_0^{\infty} \xi\varphi(\xi) J_0(r\xi) d\xi \tag{1.3}$$

We find the asymptotic representation for $\sigma_z(r, 0)$ for $r \to a+0$ and $r \to b-0$ separately.

The crux of the proposed method is the following. If the asymptotic representation is sought for $r \to a+0$, then it is first necessary to obtain the following kind of expression for σ_z:

$$\sigma_z(r, 0) = A_1(r) \frac{d}{dr} \int_a^r F_1(r, x)\, \psi_1(x)\, dx, \qquad a < r < b \tag{1.4}$$

Then letting r tend to $a+0$ in (1.4), we find the required asymptotic representation. Analogously, an expression of the form

$$\sigma_z(r, 0) = A_2(r) \frac{d}{dr} \int_r^b F_2(r, x)\, \psi_2(x)\, dx, \qquad a < r < b \tag{1.5}$$

permits obtaining an asymptotic representation for σ_z as $r \to b-0$.

The A_i, F_i, ψ_i $(i = 1, 2)$ in (1.4) and (1.5) are known functions.

2. Let us first consider the case when $r \to a+0$. In this case, we will seek the solution of (1.1) in the form

$$\varphi = S_{1/2,-1/2} h \tag{2.1}$$

Insofar as the author knows, such a solution has not been proposed before.

A survey of methods to solve equations such as (1.1) is contained in [11].

Substituting (2.1) into (1.1), we obtain

$$K_{-1/2,\,1/2}h = f, \qquad I_{1/2,\,-1/2}h = g \tag{2.2}$$

here $K_{\eta,\alpha}$ and $I_{\eta,\alpha}$ are Erdelyi-Cober operators ($\Gamma(x)$ is the gamma function)

$$K_{\eta,\alpha}f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)}\int_x^\infty (u^2-x^2)^{\alpha-1}u^{-2\alpha-2\eta+1}f(u)\,du, \quad \alpha>0$$

$$K_{\eta,\alpha}f(x) = -\frac{x^{2\eta-1}}{\Gamma(1+\alpha)}\frac{d}{dx}\int_x^\infty (u^2-x^2)^{\alpha}u^{-2\alpha-2\eta+1}f(u)\,du, \quad -1<\alpha<0$$

$$I_{\eta,\alpha}f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^x (x^2-u^2)^{\alpha-1}u^{2\eta+1}f(u)\,du, \quad \alpha>0$$

$$I_{\eta,\alpha}f(x) = \frac{x^{-2\eta-2\alpha-1}}{\Gamma(1+\alpha)}\frac{d}{dx}\int_0^x (x^2-u^2)^{\alpha}u^{2\eta+1}f(u)\,du, \quad -1<\alpha<0$$

The relationships [10]

$$S_{\eta+\alpha,\,\beta}S_{\eta,\,\alpha} = I_{\eta,\,\alpha+\beta}, \qquad S_{\eta,\,\alpha}S_{\eta+\alpha,\,\beta} = K_{\eta,\,\alpha+\beta} \tag{2.3}$$

were used to obtain (2.2).

Solving (2.2) for h, we find

$$h = K^{-1}_{-1/2,\,1/2}f, \qquad h = I^{-1}_{1/2,\,-1/2}g \tag{2.4}$$

here $K^{-1}_{\eta,\alpha}$ and $I^{-1}_{\eta,\alpha}$ are inverse operators, where [10]

$$K^{-1}_{\eta,\alpha} = K_{\eta+\alpha,-\alpha}, \qquad I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,\,-\alpha}$$

It follows from (2.2),(2.4) and the relationships (1.2) that

$$h_1 = 0, \quad h_2 = \binom{b}{r} K^{-1}_{-1/2,\,1/2}f_2 + \binom{\infty}{b} K^{-1}_{-1/2,\,1/2}f_3 \tag{2.5}$$

$$h_3 = \binom{b}{a} I^{-1}_{1/2,\,-1/2}g_2 \qquad f_3 = \binom{\infty}{r} K_{-1/2,\,1/2}h_3$$

$$g_2 = \binom{r}{a} I_{1/2,\,-1/2}h_2$$

Here the letters in parentheses in front of the operators indicate the new limits of integration.

Substituting the fourth formula in (2.5) into the second, and the fifth into the third, we arrive at the system of equations

$$h_2 = \binom{b}{r} K^{-1}_{-1/2,\,1/2}f_2 + \binom{\infty}{r} K^{-1}_{-1/2,\,1/2}\binom{\infty}{b} K_{-1/2,\,1/2}h_3 \tag{2.6}$$

$$h_3 = \binom{b}{a} I^{-1}_{1/2,\,-1/2}\binom{r}{a} I_{1/2,\,-1/2}h_2.$$

Finally, substituting the second formula in (2.6) into the first one and performing appropriate transformations, we obtain an integral equation of the second kind

$$\frac{1-x^2}{x^2}\psi(x) = 1 - \left(\frac{2}{\pi}\right)^2\int_\varepsilon^1 K(x,y)\,\psi(y)\,dy \tag{2.7}$$

$$K(x,y) = \frac{1}{2(x^2-y^2)}\left(\frac{1-y^2}{y}\ln\frac{1+y}{1-y} - \frac{1-x^2}{x}\ln\frac{1+x}{1-x}\right)$$

Here

$$\varepsilon = a / b, \quad r = bx, \quad \psi(x) = h_2^*(bx) \tag{2.8}$$
$$h_2^*(r) = -\frac{\pi^{1/2}(1-\nu)}{2\mu\delta} r^2 (b^2 - r^2)^{-1/2} h_2(r)$$

The kernel $K(x, y)$ of the integral equation (2.7) is bounded at all points of the square $\varepsilon \leqslant x \leqslant 1$, $\varepsilon \leqslant y \leqslant 1$ with the exception of the point $x = y = 1$, where it has a logarithmic singularity. In the neighborhood of the point $x = y = 1$

$$K(x, x) \approx -(1/2) \ln(1 - x)$$

The integral equation (2.7) is a Fredholm type equation since

$$\int_\varepsilon^1 \int_\varepsilon^1 K^2(x, y)\, dx dy < \infty$$

Further, using (2.8) and the last formula in (2.5), we find

$$g_2(r) = -\frac{2\mu\delta}{\pi(1-\nu) r} \frac{d}{dr} \int_a^r \left(\frac{b^2 - u^2}{r^2 - u^2}\right)^{1/2} h_2^*(u)\, du, \quad a < r < b$$

It follows from (1.1) and (1.3) that $g(r) = \sigma_z(r, 0)$. Hence

$$\sigma_z(r, 0) = -\frac{2\mu\delta b}{\pi(1-\nu) r} \frac{d}{dr} \int_\varepsilon^{r/b} \left(\frac{1 - y^2}{r^2/b^2 - y^2}\right)^{1/2} \psi(y)\, dy, \quad a < r < b \tag{2.9}$$

The force acting on the stamp is

$$P = -2\pi \int_a^b \sigma_z(r, 0)\, r dr$$

Substituting the expression for $\sigma_z(r, 0)$ from (2.9), here, we obtain

$$\delta = \gamma \frac{P(1-\nu)}{4\mu b}, \quad \gamma^{-1} = \int_\varepsilon^1 \psi(y)\, dy \tag{2.10}$$

On the basis of (2.9) and (2.10) we have the final formula to determine the stress σ_z on the contact area

$$\sigma_z(r, 0) = -\frac{\gamma P}{2\pi r} \frac{d}{dr} \int_\varepsilon^{r/b} \left(\frac{1 - y^2}{r^2/b^2 - y^2}\right)^{1/2} \psi(y)\, dy, \quad a < r < b \tag{2.11}$$

It is easy to obtain an asymptotic representation for $\sigma_z(r, 0)$ as $r \to a + 0$. Letting r tend to $a + 0$ in (2.11), we find

$$\sigma_z(r, 0) \approx -\frac{P\omega_a(\varepsilon)}{2^{3/2}\pi b^2} \left(\frac{r}{b} - \varepsilon\right)^{-1/2}, \quad r \to a + 0 \tag{2.12}$$
$$\omega_a(\varepsilon) = \gamma \varepsilon^{-3/2} (1 - \varepsilon^2)^{1/2} \psi(\varepsilon)$$

Formulas (2.12) yield a solution of the problem posed as $r \to a + 0$, i.e. upon approaching the inner contour of the stamp.

In order to evaluate the quantity $\omega_a(\varepsilon)$ it is necessary to know the function $\psi(x)$, the solution of the Fredholm equation of the second kind (2.7). The characteristic singularity of the kernel of the integral equation (2.7) is that it is independent of the parameter $\varepsilon = a / b$. The solution of (2.7) is successfully obtained in closed form only for the case $\varepsilon = 0$ (i.e. for a circular stamp). In this case

$$\psi(x) = \pi^{-1} x (1 - x^2)^{-1/2} \ln \frac{1 + x}{1 - x}, \quad \varepsilon = 0$$

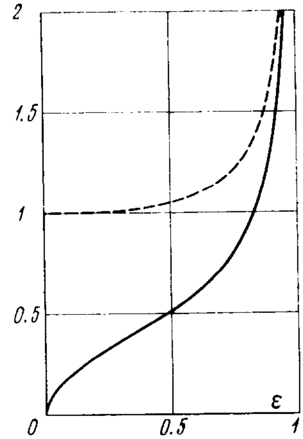

Fig. 1

For other values of ε numerical methods were relied upon for the solution of (2.7). The integral in (2.7) was replaced by a Gaussian quadrature formula (with 40 nodes). The system of linear algebraic equations obtained was solved by using a computer. A graph of the change in the quantity ω_a (Fig. 1), represented by the continuous line was constructed on the basis of these calculations.

3. Furthermore, let us consider the case when $r \to b - 0$. We now seek the solution of (1.1) in the form

$$\varphi = S_{0,\,-1/2}\,t \tag{3.1}$$

The expression (3.1) is a particular case of the solution proposed in [12]. Substituting (3.1) into (1.1), we obtain

$$I_{0,\,1/2}\,t = f, \qquad K_{0,\,-1/2}\,t = g \tag{3.2}$$

The solution procedure is carried out analogously to Sect. 2. We consequently arrive at the following formula

$$\sigma_z(r, 0) = \frac{\gamma P}{2\pi r}\frac{d}{dr}\int_{r/b}^{1}\left(\frac{y^2-\varepsilon^2}{y^2-r^2/b^2}\right)^{1/2}\eta(y)\,dy, \quad a < r < b \tag{3.3}$$

The function $\eta(x)$ satisfies the Fredholm integral equation of the second kind

$$\frac{x^2-\varepsilon^2}{x^2}\eta(x) = 1 - \left(\frac{2}{\pi}\right)^2\int_{\varepsilon}^{1}K(x, y)\,\eta(y)\,dy \tag{3.4}$$

$$K(x, y) = \frac{1}{2(x^2-y^2)}\left(\frac{x^2-\varepsilon^2}{x}\ln\frac{x+\varepsilon}{x-\varepsilon} - \frac{y^2-\varepsilon^2}{y}\ln\frac{y+\varepsilon}{y-\varepsilon}\right)$$

Formulas (3.3) and (3.4) agree with the corresponding results in [9]. Setting $r \to b - 0$ in (3.3), we obtain the asymptotic representation of the normal stress $\sigma_z(r, 0)$

$$\sigma_z(r, 0) \approx -\frac{P\omega_b(\varepsilon)}{2^{3/2}\pi b^2}(1 - r/b)^{-1/2}, \qquad r \to b - 0 \tag{3.5}$$

$$\omega_b(\varepsilon) = \gamma(1-\varepsilon^2)^{1/2}\eta(1)$$

Numerical methods were also applied to solve the integral equation (3.4). Consequently, a graph of the change in the quantity ω_b as a function of ε, shown dashed in Fig. 1, is constructed.

Formulas (2.12) and (3.5) set the nature of the singularity in the normal stress σ_z under an annular stamp upon approaching the inner and outer contours, respectively.

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STABILITY OF CIRCULAR CYLINDRICAL SHELLS OF VARIABLE THICKNESS FOR A BENDING STATE OF STRESS

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The stability problem of circular cylindrical shells of variable thickness under axial compression is examined, taking account of the bending stress of the initial pre-critical state.

The initial bending equilibrium states of shells of variable thickness are described by nonlinear differential equations, and then a linearized system of stability differential equations with variable coefficients is obtained on the basis of [1, 2]. The variable coefficients reflect the influence of the initial bending state and the variability of the shell thickness. The nonlinear equations of the pre-critical state are solved by the small parameter method for an initial axisymmetric equilibrium mode. An iteration process to determine the critical forces is constructed